

## CHAPTER 3

# The Harmful Effects of Algorithms

In Chapters 1 and 2 we have demonstrated that children should reinvent arithmetic because (1) logico-mathematical knowledge is the kind of knowledge that each child *can* and *must* construct from within, through his or her own thinking, and (2) children have to go through a constructive process similar to our ancestors' if they are to understand today's algorithms. Our third rationale for saying that children have to invent their own procedures is that the teaching of algorithms in the primary grades is harmful for the following reasons:

1. Algorithms force children to give up their own numerical thinking.
2. They "unteach" place value and hinder children's development of number sense.
3. They make children dependent on the spatial arrangement of digits (or paper and pencil) and on other people.

This chapter elaborates on each of the preceding statements and presents the data that led us to our conclusion.

### CHILDREN GIVE UP THEIR OWN NUMERICAL THINKING

When children are not taught any algorithms and are, instead, encouraged to invent their own procedures, their thinking goes in a different direction from the algorithms they are taught. For example, in addition, subtraction, and multiplication, the algorithms specify to proceed from right to left, but children's initial inventions *always* go from left to right. In division, on the other hand, the algorithm goes from left to right, but third graders' own thinking *always* goes from right to left. Figure 3.1 shows examples of what children invent for each of the four operations.

It is clear from these examples that when children are made to follow algorithms, they have to give up their own ways of thinking numerically. Because a compromise is not possible between going from left to right and going from right to left, children obey teachers by giving up their own thinking. This is in itself a sufficient reason for saying that algorithms are harmful to children.

**FIGURE 3.1** Procedures invented by children for the four arithmetical operations.

<u>18</u> <u>+17</u>	10 + 10 = 20	10 + 10 = 20	10 + 10 = 20
	8 + 7 = 15	8 + 2 = another ten	7 + 7 = 14
	20 + 10 = 30	20 + 10 = 30	14 + 1 = 15
	30 + 5 = 35	30 + 5 = 35	20 + 10 = 30
			30 + 5 = 35
<u>44</u> <u>-15</u>	40 - 10 = 30	40 - 10 = 30	40 - 10 = 30
	4 - 5 = 1 below 0	30 - 5 = 25	30 + 4 = 34
	30 - 1 = 29	25 + 4 = 29	34 - 5 = 29
<u>135</u> <u>x 4</u>	4 x 100 = 400	4 x 100 = 400	
	4 x 30 = 120	4 x 35 = 70 + 70 = 140	
	4 x 5 = 20	400 + 140 = 540	
	400 + 120 + 20 = 540		
<u>23</u> ) 285	23 + 23 + 23 + 23 . . . until the total comes close to 285		
	46 + 46 + 46 + 46 . . . until the total comes close to 285		
	10 x 23 = 230, and then proceeding by addition until the total comes close to 285		

### CHILDREN FORGET ABOUT PLACE VALUE AND DEVELOP POOR NUMBER SENSE

When children use the traditional algorithm to solve problems such as

$$\begin{array}{r} 987 \\ +345 \\ \hline \end{array}$$

they forget about place value and start by thinking and saying, for example, "Seven and five is twelve. Put two down and carry one (or ten). One and eight and four is thirteen. Put three down and carry one (or ten). One and nine and three is thirteen." The algorithm is convenient for adults, who already know place value. For primary-age children, who have a tendency to think about every column as ones, however, the algorithm serves to reinforce this weakness.

By contrast, if children are encouraged to invent their own procedures, they think and say, "Nine hundred and three hundred is one thousand two hundred. Eighty and forty is one hundred twenty; so that's one thousand three hundred twenty. And twelve more is one thousand three hundred thirty-two." The children who are allowed to do their own thinking thus strengthen and extend their knowledge of place value by using it.

The harmfulness of algorithms became evident from data obtained at Hall-Kent School in two kinds of situations. One was individual interviews of students, both those who had and those who had not been taught these rules. The other was observation in classrooms of constructivist teachers. Although most of the teachers at Hall-Kent School followed constructivist principles, some taught algorithms. Below is their distribution in 1989-91, when the following data were collected. It can be seen that algorithms tended to be taught more as the children grew older.

*Kindergarten:* None of the four teachers

*First grade:* None of the four teachers

*Second grade:* One of the three teachers

*Third grade:* Two of the three teachers

*Fourth grade:* All four teachers

All the classes were heterogeneous and comparable because the principal mixed up all the children at each grade level and divided them as randomly as possible before each school year. The transfer students from other schools were also distributed randomly among all the classes. Most of these transfer students could get right answers by using algorithms but had enormous difficulty with place value, as can be seen in Chapter 11.

**Interview Data**

In individual interviews in May 1990, the second graders were shown a sheet on which 19 computation problems were written. The children were asked to solve each problem without paper and pencil, give the answer, and explain how they got the answer. The interviewer took notes on what each child said.

*Addition.* Most of the problems presented in the interview did not yield large and consistent differences, especially when they were presented in vertical form. One of the problems,  $7 + 52 + 186$ , was presented twice in the interview, once in vertical form and later in horizontal form. The three groups of second graders did not differ very much on the vertically written questions, but striking differences emerged when the same problem was presented horizontally.

The answers given by the three classes are summarized in Table 3.1. The teacher of the first class (labeled "Algorithms" in Table 3.1) taught algorithms, but the teachers of the other two classes did not. The two classes differed, however, in that only the teacher who had the class labeled "No algorithms" immediately called parents when children were coached at home.

Most of the children in the "No algorithms" class typically began by saying, "One hundred eighty and fifty is two hundred and thirty." This is why nearly four times as many children in the "No algorithms" class got the correct answer as those in the class labeled "Algorithms" (45% compared to 12%). (The "—"s in all the tables in this chapter indicate children who did not even try to compute an answer and merely said "I can't do it," "I don't know," "I need a pencil to do it," "We haven't had this kind in class," "I forgot what the teacher said," and so on.)

The important difference, however, lay in the *incorrect* answers the children gave. The dotted lines in Tables 3.1, 3.2, and 3.3 were drawn to highlight the unreasonably large and small incorrect answers the children gave. These answers revealed inadequate knowledge of place value and poor number sense. For example, two children in the "Algorithms" class got 29 for  $7 + 52 + 186$ ! These children added all the digits as 1s ( $7 + 5 + 2 + 1 + 8 + 6 = 29$ ). Those who gave answers in the 900s did this by adding 7 to the 1 of 186 and carrying 1 from the 10s column. All the incorrect answers of the "Algorithms" class fell in the range above and below the dotted lines.

The class labeled "Some algorithms" came out between the other two. The percentage getting the correct answer was 26, which was between 12% and 45% of the other classes. The range of incorrect answers was not as outlandish as in the "Algorithms" class but not as reasonable as in the "No algorithms" class, where only two outlandish answers were given, i.e., 617 and 138.

In May 1991, a year later, an almost identical problem,  $6 + 53 + 185$ , was given to all the third- and fourth-grade classes. The results of these interviews

**TABLE 3.1** Answers to  $7 + 52 + 186$  given by three classes of 2nd graders in May, 1990 (dashes indicate that the child declined to try to work the problem)

Algorithms n=17	Some algorithms n=19	No algorithms n=20
Percentage with correct answer		
12	26	45
Incorrect answers		
9308		
1000		
989		
986		
938	989	
906	938	
838	810	
295	356	617
. . . . .		
		255
		246
		243
		236
		235
. . . . .		
200	213	138
198	213	—
30	199	—
29	133	—
29	125	—
—	114	
	—	
	—	
	—	

are summarized in Table 3.2 for third grade and in Table 3.3 for fourth grade.

All the columns except one in Tables 3.2 and 3.3 are labeled "Algorithms," indicating that the teachers of almost all third- and fourth-grade classes taught algorithms. The third column of Table 3.2, labeled "No algorithms," refers to the class taught by Sally Livingston. Although there were 22 children in her class, only 10 of them were included in Table 3.2 because only these 10 had never been taught any algorithms. The other 12 were taught these rules either at Hall-Kent School or at other schools from which they transferred.



columns. Algorithms thus appear to foster the mechanical, mindless processing of isolated columns.

By fourth grade, we expect children at least to be bothered if they add 6, 53, and 185 and get answers greater than 400 or smaller than 200. However, 39% of all the fourth graders were undisturbed by such outlandish totals, ranging from 445 to 1,215 and from 134 to 194. Nineteen percent did not even try to add the three numbers. The fourth graders who were taught algorithms for one to four years can thus be said to have done considerably worse than the second graders who were not taught these rules.

**Subtraction.** The addition problem discussed so far was written horizontally and was, therefore, difficult for the children who were used to algorithms. One of the other problems given in the interview in May 1991 was the following subtraction problem written vertically:

$$\begin{array}{r} 504 \\ -306 \\ \hline \end{array}$$

Most of the second and third graders who invented their own procedures said, "Five hundred take away 300 is 200. Four take away 6 is 2 less than zero; so the answer is 198." The percentages of children in the "No algorithms" classes who thus gave the correct answer were 74% in second grade ( $n = 19$ ) and 80% in third grade ( $n = 10$ ). The wrong answers they gave were not far from 200. They were 320, 202, 202, 200, and 194 in second grade and 202 and 190 in third grade.

The percentages in the "Algorithms" classes who gave the correct answer were only 42% and 35% in third grade and 55%, 39%, 38%, and 29% in fourth grade. These percentages were all lower than those of the "No algorithms" classes in second and third grade. The incorrect answers given by the third- and fourth-grade "Algorithms" classes are summarized in Table 3.4. It can be observed in this table that the ranges of wrong answers were again enormous and much greater than in the "No algorithms" classes.

As stated earlier, vertically written problems are usually much easier for children who are used to following algorithms. In this subtraction problem, however, the vertical arrangement did not help the "Algorithms" children. Because their knowledge of place value was poor, many of them got the answer of 108 by borrowing 10 from the 5 of 504 and subtracting 3 from 4. Others got 208 by adding 10 to 4 without borrowing it from anywhere! In Table 3.4, the large variety of answers ending with an 8 is striking, indicating that the children had learned to subtract 8 from 14 without knowing where the 10 came from.

The poor number sense of children who are taught algorithms is caused not only by inadequate knowledge of place value but also by the habit of thinking only about isolated columns. This habit is especially evidenced by the children

504  
**TABLE 3.4** Incorrect answers to  $\begin{array}{r} 504 \\ -306 \end{array}$  given in May, 1991 by 3rd and 4th graders who were taught algorithms. (Dash indicates that the child declined to try to work the problem)

Third grade classes		Fourth grade classes			
n=19	n=20	n=20	n=21	n=21	n=18
			898		
			808		
			498	308	
			298	298	
	1106		298	298	
	708		298	208	
	298	410	208	208	408
406	207	208	208	205	208
.....	.....	.....	.....	.....	.....
202			199	202	202
196			194	196	196
194				192	192
192					
.....	.....	.....	.....	.....	.....
108	164	189	189	148	108
106	113	108	189	108	108
"8, 0, 2"	109	108	108	108	"8, 0, 1"
"8, 0, 2"	108	108	108	"8, 0, 2"	"8, 0, 1"
"8, 10, 1"	108	19	108		"2, 0, 2"
"2, 0, 2"	108	"8, 0, 2"			-
	108	"2, 0, 2"			
	108				
	22				

in Table 3.4 who gave answers such as "Eight, zero, one" for the columns, from right to left. If these children only *thought* about whole numbers, and subtracted *about* 300 from *about* 500, they would know that the answer *has to be* about 200.

When second and third graders can perform so much better than fourth graders, we must conclude that there is something seriously wrong with using algorithms in the early grades.

**Multiplication.** The multiplication of a two-digit number by a two-digit number, such as  $13 \times 11$ , is not introduced in the textbook before fourth grade. This problem was, therefore, in a sense unfair for third graders because only the "No algorithms" class had been exposed to this kind of problem. This problem was nevertheless given to find out about children's number sense and their ability

to invent solutions when faced with an unfamiliar problem. Some of the second graders in constructivist teachers' classrooms solve multiplication problems by addition—by adding 13 and 13 (for two 13s), doubling the result (for four 13s), adding 52 and 52 (for eight 13s), and adding 39 (for three more 13s) to 104.

Sixty percent of the third graders who had never been taught algorithms gave the correct answer to  $13 \times 11$ . By contrast, only 11% and 5% of the third-grade "Algorithms" classes gave the answer of 143. The important difference, however, lay in the incorrect answers the three groups produced. In the "No algorithms" class, the incorrect answers were: 79, 113, 146, and "I want to skip it." In the two "Algorithms" classes combined, the incorrect answers were: 13, 13, 13, 13, 23, 33, 33, 42, 93, 113, 113, 130, 131, 131, 133, 133, 133, 133, 133, 133, 133, 155, 330, 1013, and 1133, and 10 children refused to try. It can be said again that the range of wrong answers in the "Algorithms" classes revealed poor number sense.

All the fourth graders could easily get the correct answer to this problem by using a pencil and following the algorithm. When they were allowed to use only their minds, however, the percentages getting the correct answer were only 5%, 6%, 14%, and 15% for each of the four classes. The incorrect answers produced by all four of the fourth-grade classes combined were: 11, 13, 13, 13, 13, 13, 13, 13, 23, 23, 23, 26, 26, 26, 26, 33, 34, 42, 42, 44, 44, 44, 44, 45, 45, 64, 66, 113, 123, 131, 133, 133, 133, 133, 133, 133, 133, 133, 133, 133, 135, 140, 141, 141, 141, 144, 153, 443, 1300, 1313, 1313, and 1326. (Twenty children refused to try.)

Examples of some of the ways in which these answers were obtained are the following: "Thirteen times 1 is 13, and 13 times another 1 is 13. So the answer is thirteen thirteen. One, three, one, three." "Thirteen times 10 is 130, plus 1 is 131." "Thirteen times 1 is 13. Put down the 3 and carry the 1. Thirteen times 1 is 13, so the answer is 133." "One times 3 is 3. One times 3 is another 3, and 1 times 1 is 1. So the answer is 133."

### Classroom Observation Data

As stated earlier, the performance of students who had been taught algorithms before coming to classrooms of constructivist teachers also convinced us of the harmful effects of this instruction. These children could usually get right answers by using algorithms but had enormous difficulty with place value, as can be seen in Chapter 11. We let them use whatever procedure they chose as long as they could explain their steps.

Under the pressure of having to explain their procedures, above-average transfer students quickly conclude that the left-to-right method used by their classmates is easier. By contrast, average students continue to use algorithms and eventually come to understand place value. The below-average students,

however, cling tenaciously to algorithms without much progress in knowledge of place value. Algorithms provide the security of producing correct answers, and below-average students continue to function like machines that cannot be unprogrammed. Their thinking remains blocked and paralyzed by the program.

The harmful effects of algorithms became even more evident when one of the fourth-grade teachers, Cheryl Ingram, decided to try the constructivist approach in 1991–92. After 10 years of teaching fourth grade, Cheryl decided to change her teaching because the children who had been in constructivist classes for one, two, or three years seemed to be better math students. I (CK) sat in her class almost every day during the math hour throughout the year to help her become a constructivist teacher and was amazed by the difficulty of unprogramming fourth graders. The following account indicates the extent to which the children did not understand place value and treated isolated columns from right to left.

One of the ways in which Cheryl tried to wean children away from algorithms was to write problems such as  $876 + 359$  horizontally on the chalkboard and ask the class to invent a variety of ways to solve them without using a pencil. As the children volunteered to explain how they got the answer of 1,235 by using the algorithm in their heads, she followed their statements and wrote numbers such as the following for each column:

$$\begin{array}{r} 15 \\ 13 \\ +12 \\ \hline 40 \end{array}$$

After the child finished explaining how he or she got the answer of 1,235, Cheryl said, "But I followed your way and got 40 as my answer. How did you get 1,235?" Most of the children were stumped and became silent. However, one child soon pointed out that the teacher's 13 was really 130, and that her 12 stood for 1,200.

This kind of place-value problem was relatively easy to cure. The difficulty that persisted was the column-by-column approach that prevented children from thinking about whole numbers. Presented with problems such as the preceding addition problem, the children continued to give fragmented answers from right to left, such as "5 [for  $6 + 9$ ], 130 [for  $10 + 70 + 50$ ], and 1,200 [for  $100 + 800 + 300$ ]."

In an effort to get children to think about whole numbers, we conducted an experiment on October 28, 1991. Cheryl put on the chalkboard one problem after another, such as the following, that had 99 in one of the addends:  $366 + 199$ ,  $493 + 99$ , and  $601 + 199$ . During the entire hour, Cheryl gave only this kind of problem for the class to solve in many different ways.

Almost all the children in the class continued to use the algorithm during the

entire hour and added the 1s first, carried 10, added the 10s, and then carried 100. One of the children, however, whom we will call Joe, had been in constructivist classes since first grade and volunteered solutions like the following for every single problem: "I changed  $366 + 199$  to  $200 + 365$ , and my answer is 565." After an entire hour of this kind of "interaction," only three children in the class were imitating Joe! The rest of the class continued to deal with each column separately.

In the meantime, by mid-October, Cheryl had remarked that in her 10 years of teaching fourth grade, she had never seen such excitement and enthusiasm for math. In early November, she felt the need to announce that the class *had* to invent ways of adding and subtracting without carrying and borrowing. This requirement sparked some creativity in the class, and one of the formerly passive students began to wave her hand with confidence. On November 19, she invented the following solution for  $606 - 149$  that indicated her thinking about whole numbers:

$$\begin{aligned} 600 - 100 &= 500 \\ 6 - 49 &= \text{negative } 43 \\ 500 - 43 &= 457 \end{aligned}$$

There were many ups and downs throughout the year, and December 20 brought a disappointment. Cheryl told the class that she had \$50 to spend on Christmas presents and wanted to know if she had enough money to buy the items listed below:

3 <i>Battleships</i> (a game)	@ 7.99
2 sweaters	@ 11.99
1 wallet	15.00
2 dolls	@ 8.95

The first volunteer started her answer by saying, "Nine plus 9 plus 5 equals 23."

January 21 brought the first left-to-right procedure from users of algorithms. Cheryl put the following prices on the board and asked the class for the total amount:

Shirt	\$5.00
T-shirt	1.95
Sweater	37.90

As usual, one student asked, "Can I start on the right?" Andrew immediately reacted by saying, "It's easier to start on the left." Rob quickly agreed. Andrew explained that  $37 + 1 + 5$  made 43 dollars, that 90 cents and 10 cents made another dollar, and that the answer was 44 dollars and 85 cents.

January 29 however, brought a disappointment. Cheryl wrote the following

misaligned numbers on the board, and Andrew volunteered the answer of 160 as shown below it:

$$\begin{array}{r} 25 \\ 3 \\ 4 \\ \hline +65 \\ \hline 20 + 30 = 50 \\ 40 + 60 = 100 \\ 150 + 10 = 160 \end{array}$$

When Cheryl asked who agreed with Andrew, five hands went up.

The children nevertheless made considerable progress by the end of the year, and the interviews conducted in May 1992 produced much better results than in 1991. The percentage giving the correct answer to  $6 + 53 + 185$  increased from 17 in 1991 (see the last column of Table 3.3) to 75 in 1992. The range of errors also decreased to 28, 202, 234, 238, and 243. Figure 3.2 shows the relationship between the use of the algorithm and the frequency of getting the correct answer. It can be seen in this figure that 13 (76%) of the 17 children in Cheryl's class used the algorithm in 1991 and got incorrect answers. In 1992, by contrast, 15 (75%) of the 20 children used invented methods and got the correct answer. This analysis indicates that children are more likely to get the correct answer if they do their own thinking.

As for the subtraction problem,  $504 - 306$  written vertically, the percentage giving the correct answer increased from 39% (in 1991) to 80% (in 1992). The incorrect answers in 1992 were 90, 108, 200, and 202. These were much more reasonable than the errors of the previous year, which can be seen in the last column of Table 3.4. As can be seen in Figure 3.3, all the children used the conventional algorithm in 1991, and only 7 of the 18 children (39%) got the correct answer. In 1992, by contrast, 16 of the 20 children used invented procedures, and 15 of them got the correct answer. This analysis again demonstrates that children who do their own thinking are more likely to get the correct answer.

The percentage giving the correct answer to  $13 \times 11$  increased from 6% (in 1991) to 55% (in 1992). The wrong answers produced each year were the following:

1991: 11, 13, 42, 64, 113, 133, 133, 141, 144 (with eight children refusing to try)  
1992: 113, 133, 144, 233, 300

While the data looked much better in 1992, these fourth graders cannot be said to have overcome the damage caused by algorithms. In class, many of them

FIGURE 3.2 The relationship between using the conventional algorithm and getting the correct answer to  $6 + 53 + 185$ .

1991\*

	Algorithm	Invented procedures
Correct answer	3	0
Incorrect answer	13	1

\*One child was excluded from this analysis because she said she was thinking of multiplying 185 by 53 and of adding 6.

1992

	Algorithm	Invented procedures
Correct answer	0	15
Incorrect answer	2	3

continued to approach every addition and subtraction problem mechanically and to think about each column separately. The cognitively most advanced children came close to being unprogrammed by the end of the school year. The below-average students, however, continued to cling to algorithms and to have trouble with place value. Human beings are much harder to unprogram than computers, and children at the bottom of the class suffer the most from the damage caused by algorithms.

FIGURE 3.3 The relationship between using the conventional algorithm and getting the correct answer to  $\frac{504}{-306}$ .

1991

	Algorithm	Invented procedures
Correct answer	7	0
Incorrect answer	11	0

1992

	Algorithm	Invented procedures
Correct answer	1	15
Incorrect answer	3	1

### CHILDREN BECOME DEPENDENT ON THE SPATIAL ARRANGEMENT OF DIGITS AND ON OTHER PEOPLE

In interviews, children in "Algorithms" and "No algorithms" classes gave different reasons for not *trying* to compute an answer. Most of the reasons given in "Algorithms" classes were: "I need a pencil," "We haven't had this kind yet," or "I can't remember what the teacher said." While these students revealed their dependence on pencil and paper, the spatial arrangement of digits, and other people, children who have never been taught algorithms said, "I can't do



it," "I don't know how," or something else that expressed their own inability to compute the answer.

Some children in constructivist classes indeed cannot solve certain problems. However, these children have at least not learned to be dependent on paper and pencil, the spatial arrangement of digits, and other people to solve problems. Algorithms enable children to produce correct answers, but the side effect is the erosion of self-reliance.

## CONCLUSION

Algorithms and "alternative" or "informal" methods have been discussed in a variety of ways for many years. Some people have advocated teaching algorithms *and* encouraging "alternative" methods (Lankford, 1974; National Council of Teachers of Mathematics, 1989). Others in Brazil (Carraher, Carraher, & Schliemann, 1987; Carraher & Schliemann, 1985) and England (Jones, 1975) have questioned the desirability of teaching algorithms. A third group has urged, from a variety of perspectives, that we stop teaching algorithms. This position can be found not only in the United States (Burns, 1992a, 1992/93; Madell, 1985) but also in Denmark (Bennedbek, 1981), England (Plunkett, 1979), Holland (Treffers, 1987), and South Africa (Murray & Olivier, 1989; Murray, Olivier, & Human, 1992; Olivier, Murray, & Human, 1990, 1991). We agree with the third group but go a step further by saying that the teaching of algorithms is harmful to children in the primary grades.