Chapter 1

Multiplication

1.1 Early methods

We shall begin by describing the method known to some as "Egyptian multiplication" and to others as "Russian multiplication". Evidently, there are many claimants for this method! We shall illustrate the method with the computation of 11×17. Two rows are prepared as shown below; successive halving takes place in one row (giving the numbers 11, 5, 2, 1; observe that remainders are not retained) and successive doubling in the other (giving the numbers 17, 34, 68, 136).

HALVING	11	5	2	1
DOUBLING	17	34	68	136

Scanning through the HALVING row, we identify the entries which are *odd*, and compute the sum of the corresponding entries in the DOUBLING row. Here, we compute the sum $17 + 34 + 136$, as these are the numbers that correspond to 11, 5 and 1 in the HALVING row. The sum is 187, and this is the product we seek ($11 \times 17 = 187$).

Similarly, to compute 47×91, we obtain a sequence of halves (47, 23, 11, 5, 2, 1; as earlier, remainders are not retained), and a sequence of doubles (91, 182, 364, 728, 1456, 2912). The only even number in the first sequence is 2, so we compute the sum $91 + 182 + 364 + 728 + 2912 = 4277$. Thus, $47 \times 91 = 4277$.

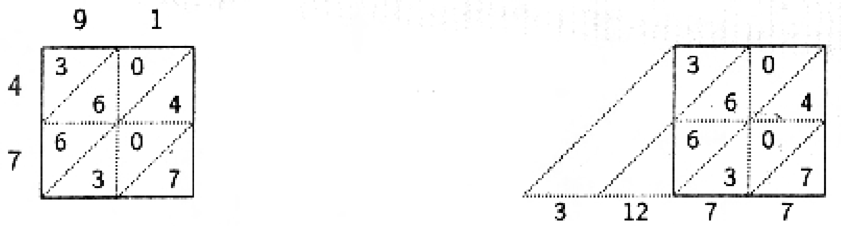

Figure 1.1. *Napier's grid*

Exercises

1.1.1 Compute 23×43 and 231×311 using this method.

1.1.2 Consider the computation of 16×103; let 16 be the number which is halved. What is the HALVING sequence in this case?

1.1.3 Consider the computation of 64×103; let 64 be the number which is halved. What happens in this case?

1.1.4 In the multiplication of 2-digit numbers by this technique, we find that we never need to add more than 7 numbers together. Why is this so?

1.1.5 Explain why the method works for the case when the number which we halve is of the form $2 \times 2 \times 2 \times \cdots \times 2$ (that is, a product of twos).

1.1.6 Explain why the method works in general.

1.2 Napier's bones

Another method is the 'grid' technique which probably originated in India early in the tenth century. We illustrate the technique with the computation of 47×91. The products of the individual digits are $4 \times 9 = 36$, $7 \times 9 = 63$, $4 \times 1 = 4$ and $7 \times 1 = 7$. The products go into an array as shown in Figure 1.1.

Reading down along the upward-sloping diagonals, left to right, the sums are 3, 12, 7 and 7. Working from right to left, we see that the required product is 4277 (the '1' in 12 has been carried and added to 3, giving 4). The transition from this technique to the one we currently use in schools to do long multiplication should be quite clear.

Centuries later, John Napier of Scotland patented a set of wooden rods that helped in these calculations. The device came to be known later as "Napier's bones".

Exercises

1.2.1 Compute 1235×571 using the grid method.

1.2.2 Make a set of 'Napier's bones' for yourself and use the set to do long multiplication.

1.3 Multiplication tables

Everyone is familiar with multiplication tables: $9\times 1 = 9$, $9\times 2 = 18$, $9\times 3 = 27, \ldots$. However, tables of this sort have a serious limitation to them—they are just not big enough. What if one wanted to compute, say, 1234×5678? Would this require having to look up a table of products of 1234? And what about 14259×71326? It would be impossibly tedious and painful to have to build up so extensive a multiplication table as to be able to handle every conceivable multiplication—there would not be enough space to write down such a table! The early bankers, astronomers and navigators, who had to do vast numbers of hand-computations on a routine basis, certainly had an unenviable task on their hands.

Mathematicians realized a long time ago that the big difficulty with multiplication tables is that they have to be 2-dimensional; after all, multiplication needs *two* inputs. Mathematicians refer to such operations as *binary operations* ('binary' for 'two'). Suppose we needed a table for multiplication of numbers till 9. We would have to produce something like the following.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

The reader will notice at once how large the array is; and this is for numbers till 9 only! The difficulty is, of course, that the array is 2-dimensional; it has to have $9 \times 9 = 81$ entries in all. Is it possible to make do with a 1-dimensional array instead?

Quite early on, it had been found that this *is* possible: simply by preparing a table of squares. The key observation is that the product $a \times b$ can be written as a difference of squares:

$$a \times b = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

Thus, to compute 6×14, we do the following:

$$\frac{14+6}{2} = 10, \quad \frac{14-6}{2} = 4,$$
$$10^2 - 4^2 = 100 - 16 = 84.$$

So the answer is: $6 \times 14 = 84$. This works for larger numbers as well, but we need to have a table of squares to refer to. For instance, to compute 1357×9131, we first compute

$$\frac{9131 + 1357}{2} = 5244, \quad \frac{9131 - 1357}{2} = 3887,$$

then we refer to the table of squares and look up

$$5244^2 = 27499536, \quad 3887^2 = 15108769,$$

and finally we do a subtraction: $27499536 - 15108769 = 12390767$. Therefore the answer is,

$$1357 \times 9131 = 12390767.$$

The reader should appreciate the saving achieved in having to prepare a 1-dimensional array as compared to a 2-dimensional array. (Note that we need to include the squares of half-integral numbers in our list, in order to multiply odd numbers with even numbers.

EXAMPLE. To do 61×72, we look up $66.5^2 = 4422.25$ and $5.5^2 = 30.25$, and obtain the answer as $4422.25 - 30.25 = 4392$.)

This takes care of multiplication, but what do we do about division, and what about the computation of powers and roots? Clearly, the method is limited in its utility. We need something that is a lot more flexible and powerful.

EXERCISES

1.3.1 Compute 211×319 and 171×352 using this technique. (You will need to refer to a table of squares.)

1.3.2 For multiplication of 3-digit numbers with one another, how large must the table of squares be? If we had to prepare an array that gave all possible products of 3-digit numbers with one another, how large would the array have to be? What saving is achieved by going in for the table-of-squares technique?

1.4 Interlude on prime factorization

The tabulation of a table of squares has an unexpected bonus—it helps in finding the prime factorization of numbers. The idea is very simple: if an integer N can be expressed as a difference of squares, say $N = a^2 - b^2$ where $a - b$ exceeds 1, then we have $N = (a-b)(a+b)$, and we have found the sought-after factorization. We show below how the idea can be put into practice.

Let the number to be factorized be $N = 1073$. We compute the quantities $N + 1^2$, $N + 2^2$, $N + 3^2$, ..., and continue till a square number is obtained. Obviously, we shall need to refer to a table of squares at each stage. Here is what we obtain.

$$N + 1^2 = 1074, \quad N + 2^2 = 1077,$$
$$N + 3^2 = 1082, \quad N + 4^2 = 1089 = 33^2.$$

We observe that $N + 4^2 = 33^2$. It follows that

$$N = 33^2 - 4^2 = (33 - 4)(33 + 4) = 29 \times 37,$$

and we have succeeded in factorizing N.

For $N = 2201$, we find that the first square in the sequence is

$$N + 20^2 = 2201 + 400 = 2601 = 51^2,$$

and this yields the factorization $N = 51^2 - 20^2 = 31 \times 71$.

EXERCISES

1.4.1 Factorize the numbers 2117 and 2911 using this technique.

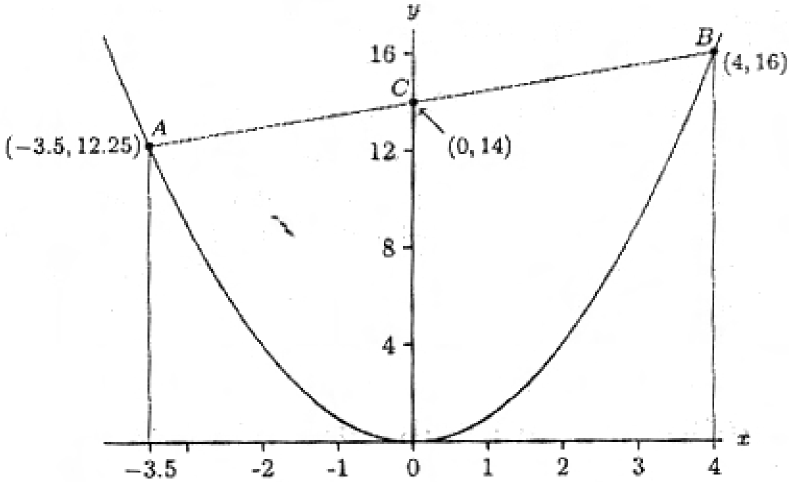

Figure 1.2. *Multiplication via the parabola* $y = x^2$

1.4.2 Find the factorization of 64777.

1.4.3 How good is the method? More precisely, which numbers require a small number of computations, and which numbers require a large number of computations?

1.5 Parabolic multiplication

We describe here a pretty, though perhaps not too practical, method for multiplying two numbers.

The method involves drawing the graph of a curve known as a *parabola*. We prepare a table of squares, then plot points with coordinates of the form (x, x^2) for numerous values of x; i.e., points with coordinates such as $(-4, 16)$, $(-3, 9)$, ..., $(0, 0)$, $(1, 1)$, ..., $(3, 9)$, $(4, 16)$, ... The resulting points fall neatly on a curve as shown in Figure 1.2. (Note that the scales are not the same: the scale on the y-axis is more compressed than the scale on the x-axis.)

Suppose we wish to multiply two numbers, say 3.5 and 4 (not much of a multiplication!—but it will do for the purposes of illustration). Points A and B are located on the curve as shown above: the x-coordinates of A and B are -3.5 and 4, respectively (note the negative sign for the x-coordinate of A), and segment

AB is drawn. Let AB meet the y-axis at C; then the y-coordinate of C yields the required product. From the sketch we see that the point C is $(0, 14)$, so the required product is 14.

More generally, to compute $a \times b$, we draw the segment connecting the points $A\,(-a, a^2)$ and $B\,(b, b^2)$; then the y-coordinate of the point C where AB meets the y-axis yields the desired product.

The reason is easy to see. The slope of segment AB is

$$\frac{b^2 - a^2}{b - (-a)} = \frac{b^2 - a^2}{b + a} = b - a,$$

so if $C = (0, c)$, then by equating the slopes of BC and AB, we get

$$\frac{b^2 - c}{b - 0} = b - a, \quad \therefore \quad b^2 - c = b(b - a), \quad \therefore \quad c = ab.$$

So the y-coordinate of C is ab; hence the result.

1.6 Interlude on nomograms

The graphical approach described above may remind us of *nomograms*. These are graphical devices used to compute the values of a two-variable function f (which by definition needs two inputs; e.g., f could be addition or multiplication). In a nomogram you have three parallel scales, each with numbers written on it in some appropriate manner. For convenience, we refer to the top and bottom scales as the A and B scales, and to the middle one as the C scale. The nature of f decides the way the scales are to be graded. The manner of usage is this: to compute $f(a, b)$, we locate the numbers a and b on the A and B scales; hold a ruler so that its edge passes through the points a and b, then read off the answer from the point where the ruler meets the C scale. Nomograms are of great use on the engineering shopfloor, where particular kinds of computations need to be done repetitively. In cases where the computations are hard to do by hand, nomograms prove to be very convenient. We shall consider a few examples to render the idea more concrete.

- *Arithmetic mean.* Let f be the AM function given by $f(a, b) = (a + b)/2$. The three scales in this case are graded uniformly, all in the same way.

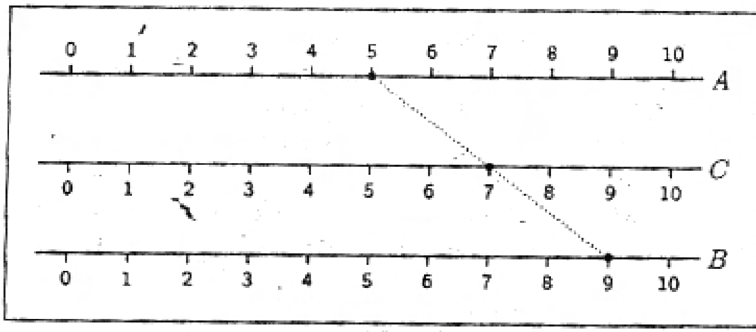

Figure 1.3. *A nomogram to compute the arithmetic mean*

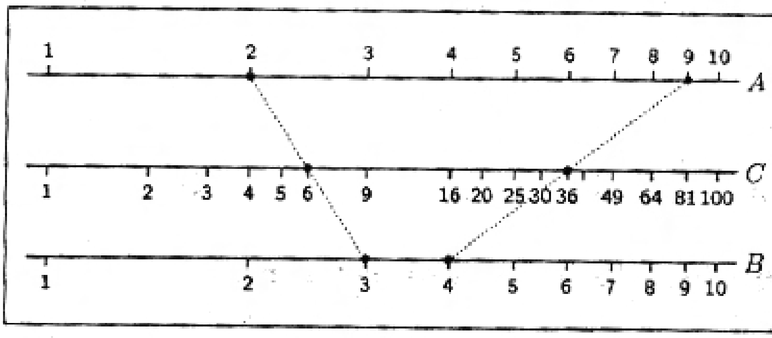

Figure 1.4. *A nomogram to compute products*

A 'ruler' has been drawn (Figure 1.3) to show how the AM of 5 and 9 is computed to be 7.

- *Product.* Let f be the product function given by $f(a, b) = ab$. The appropriate nomogram is shown in Figure 1.4. Observe that none of the scales have uniform gradation.

  A 'ruler' has been drawn to show how 2×3 is found to be 6, and how 9×4 is found to be 36.

Nomograms can be constructed to handle many kinds of two-variable functions, and their construction sometimes involves a lot of challenging and pretty mathematics.

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Perhaps the most well-known form of the nomogram – though it is not generally referred to as such – is the slide rule, which was invented during the nineteenth century and, for many generations, was considered to be an indispensable tool for every working engineer and scientist. A popular stereotype of the first half (and more) of the twentieth century is the engineer with a slide rule sticking out of his pocket. Interestingly, when the famous geneticists Jim Watson and Francis Crick posed for their Nobel prize photograph in front of their bits-and-pieces double helical model of the DNA, Crick chose to do so with a slide rule in his hand, pointing it at the model in a rather theatrical manner.

But history has its own ironies; today the slide rule is no more, and few children (and even adults) have ever seen or handled one. Such is the power of the electronic revolution!