Where is the square to complete?

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Sunita was extremely upset. She thought she had learnt something new and interesting that day, but now she was looking at it again, she could not understand anything at all!

They were working with quadratic equations in class. Until a few days ago, they only saw equations that looked like: $4x^2-20x+25 = 0$. This was an equation Sunita found very easy to solve. She could *see* that $4x^2-20x+25$ was the same as $(2x-5)^2$, so, if $(2x-5)^2 = 0$, then of course, $2x-5 = 0$ and hence $x = 5/2$.

Then they had looked at equations like: $2x^2-15x+25 = 0$. These meant more work for Sunita, but after some trial and error, she found that $2x^2-15x+25 = (2x-5)(x-5)$, which meant $(2x-5)=0$ or $(x-5)=0$. So she realised that $x=5/2$ or $x=5$.

From this, solving $2x^2-16x+30 = 0$ was easy, though it looked difficult. Sunita happily checked that $2x^2-16x+30 = (2x-6)(x-5)$ and hence $x=3$ or $x=5$.

Sunita had been working on lots of examples and was slowly trying to get an idea of how to find *factors* like in these examples. There seemed to be no specific way, one had to guess and check, which was sometimes fun, and sometimes very difficult.

That's how it was, until today.

Today in class, her teacher Mr Arun had said no guess was necessary, and instead taught them a technique called **Completing the Square**. For example, this is what one could do:

$4x^2 - 8x - 21 = 0.$

Divide by 4, so that the first coefficient is 1.

$x^2 - 2x - 21/4 = 0.$

Rewrite it as: $x^2 - 2x = 21/4$.

Take half-the coefficient of the x-term, square it and add it to both sides:

$x^2 - 2x + 1 = (21/4 + 1).$

Notice that the left hand side simplifies, while the right hand side evaluates to 25/4, so we have

$(x-1)^2 = 25/4.$

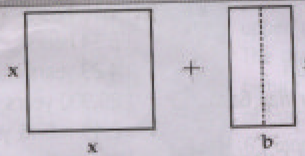
So the solutions are $x-1 = +5/2$ or $-5/2$.

Hence $x=7/2$ or $x=-3/2$.

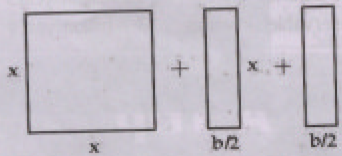
This sounded very easy, and Mr Arun then said, "*So, for any equation of the form* $x^2 + bx = c$, *we can do the same*" and then did a long sequence of derivations finally ending with, "*so, the solution is,* $x = - b/2 +$ sqrt{c+b²/4}". Here, + is to be read as "plus or minus" and sqrt as "square-root-of".

Sunita was struggling to understand the derivation. This was called "completing the square"? Which square? Why? How??

Sunita's mother saw that her little girl was looking very troubled. Once she saw what the problem was, all she did was to draw some pictures, and everything became crystal clear to Sunita. Want to see what magic Sunita's mother came up with? Follow the picture-proof below.

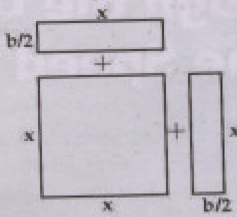
1. 

= c $(x^2 + bx = c)$

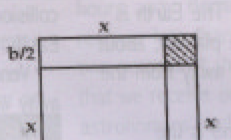
2. 

= c $(x^2 + \frac{b}{2}x + \frac{b}{2}x = c)$

(Redrawn b–piece)

3. 

= c **(Left side redrawn)**

4. 

= c +  b/2 ;
b/2

(Added small square of side b/2 on both sides)

5. **Area equation for above :** $\left(x+\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$;

6. **Taking the square root :** $x+\frac{b}{2} = +\sqrt{c+\left(\frac{b}{2}\right)^2}$ or $-\sqrt{c+\left(\frac{b}{2}\right)^2}$.

7. **Solutions :** $x = -\frac{b}{2} + \sqrt{c+\left(\frac{b}{2}\right)^2}$ or $x = -\frac{b}{2} - \sqrt{c+\left(\frac{b}{2}\right)^2}$.