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Patterns as Tools for Algebraic Reasoning

Kristen Herbert and Rebecca H. Brown

One teacher made up this story to introduce her sixth-grade students to "Crossing the River," one of the lessons in a unit called Patterns in Numbers and Shapes.

> You and your partner have gone on a long hike with 8 adults. You are very hot and tired. Your group gets to a wide river that you must cross to get home. No one in your group knows how to swim. On the riverbank is a small boat, which can only hold 2 children, or 1 adult, or 1 child. How many one-way trips does it take to get all the people in your group across the river?

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The authors are very grateful to the many teachers and students who field-tested this unit and would like to thank their colleagues at the Education Development Center, who helped create this unit, and Charles Lovitt, who provided the initial drafts of many of the problems in the unit.

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After a brief discussion of the story by the class, groups of students sought different tools to solve the problem. Some groups began diagramming the story on paper, others acted it out, and others used cubes of different colors to track the trips across the river. Whatever their strategy, students in each group were following an investigative process that they had learned for finding and generalizing patterns. At the same time, they were building a foundation in algebraic thinking.

The Patterns in Numbers and Shapes unit was developed as part of the National Science Foundation-funded curriculum MathScape: Seeing and Thinking Mathematically and was field-tested in classrooms across the country. In each lesson, students encounter a problem presented in a context, and they work in pairs or small groups to act out the story, whether kinesthetically, visually by drawing pictures, or manipulatively by modeling the situation with physical objects. They engage in an investigative process to solve the problem: (1) they seek out a pattern in the story, (2) they recognize the pattern and describe it using different methods, and (3) they generalize the pattern and relate it to the story. The Patterns in Numbers and Shapes unit and all the algebra-focused units in the curriculum are intended to provide a new image of how students can develop algebraic-thinking skills. A broad view of algebraic thinking is taken to show students the real-life uses and relevance of algebra. Algebraic thinking is using mathematical symbols and tools to analyze different situations by (1) extracting information from a situation, such as the one described in “Crossing the River”; (2) representing that information mathematically in words, diagrams, tables, graphs, and equations; and (3) interpreting and applying mathematical findings, such as solving for unknowns, testing conjectures, and identifying functional relationships, to the same situation and to new, related situations. The investigative process used in the Patterns in Numbers and Shapes unit is an initial, informal example of this three-part framework (see fig. 1).

Using this investigative process to solve contextualized problems gives students the kind of informal exploration of algebra discussed in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989, 102). It states, “It is thus essential that in grades 5–8, students explore algebraic concepts in an informal way to build a foundation for the subsequent formal study of algebra.” Rather than push students into formal symbolic algebra, the unit emphasizes algebraic thinking by leading students to communicate their toughts in their own words or their own symbols. In addition, the five-week unit gives students the opportunities and support to develop pattern-seeking skills and an ability to generalize patterns from concrete situations. The approach is also intended to increase students’ confidence in themselves as being capable algebra students.

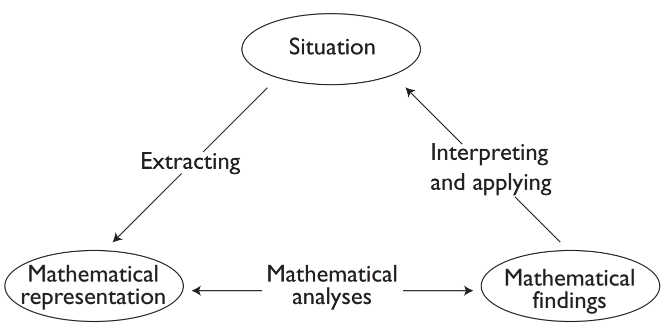

Fig. 1. A framework for algebraic thinking in context

The Investigative Process to Solve Problems

The investigative process consists of three phases: (l) pattern seeking, (2) pattern recognition, and (3) generalization. These three phases are specific components of the general framework shown in figure 1; pattern seeking is extracting information, pattern recognition is mathematical analysis, and generalization is interpreting and applying what was learned. Students follow the investigative process for all twelve lessons in the unit. “Crossing the River,” one lesson from the unit, will next be used to illustrate the investigative process that students follow. Students are guided through the lesson by the student page shown in figure 2. In a teacher’s guide, suggestions are made for the use of manipulatives, physical activity, and drawings to help students solve the problems.

1. A group of 8 adults and 2 children need to cross a river. A small boat is available that can hold 1 adult, or 1 child, or 2 children. Everyone can row the boat.

   How many one-way trips does it take for them all to cross the river?

2. What if there were different numbers of adults? How many trips would be required in the following situations?
   - 6 adults and 2 children?
   - 15 adults and 2 children?
   - 3 adults and 2 children?
   - 100 adults and 2 children?

3. How would you find the number of trips needed for 2 children and any number of adults? Describe the method you would use to solve the problem.

Fig. 2. Student sheet for “Crossing the River”

Pattern Seeking

To begin the “Crossing the River” problem, students model the situation presented in the first step of the task (see step 1 in fig. 2) to look for patterns. As suggested in the teacher’s guide, teachers give students some type of counters to help solve the task; two counters of one color represent the children, and eight of another color represent the adults (see fig. 3).

As students begin to move the boat back and forth across the river, they begin to make comments, such as, “Someone has to row the boat back across the river, maybe a kid can do it” and “The arms on these kids are going to be really tired after all this rowing!”

Students collect and record numerical data as they work through the situation. They develop many different methods of recording the data. Some stu-

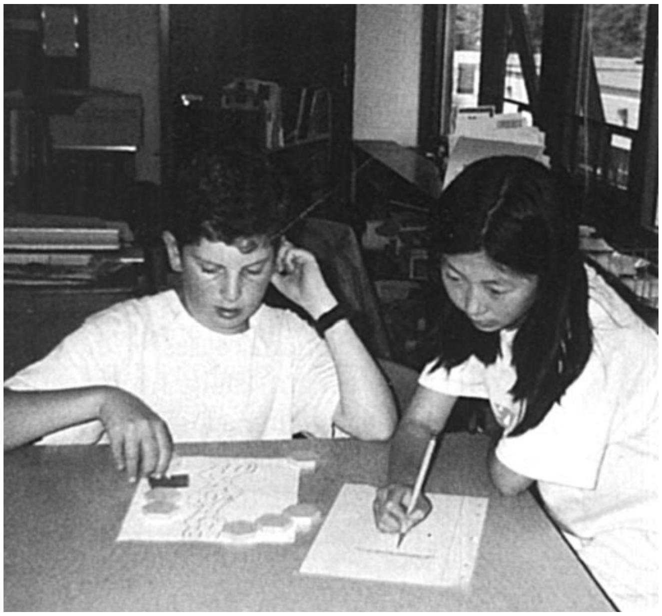

Fig. 3. Students work through the problem using manipulatives and a river drawn on paper.

dents record the number of trips either as tally marks or in a list. Others draw a picture or diagram that shows each trip across the river (fig. 4).

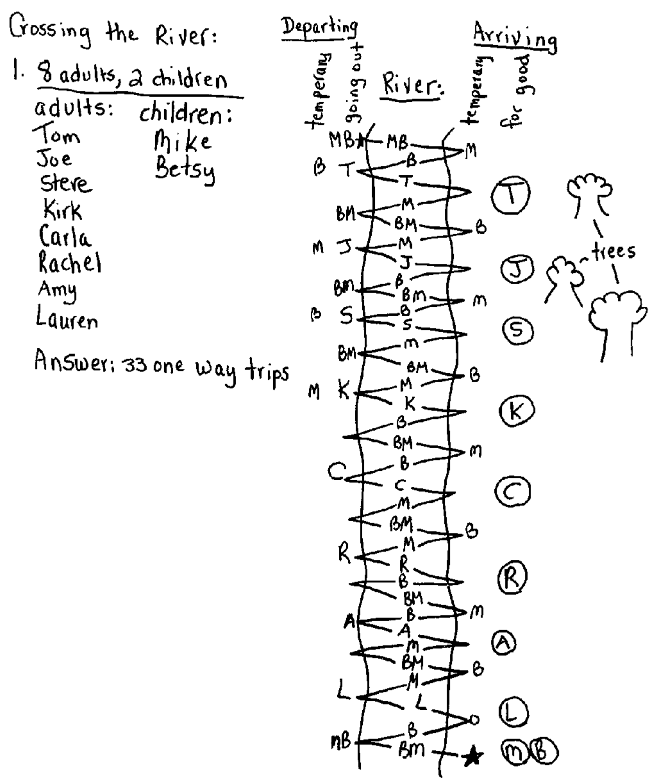

Fig. 4. One student recorded the trips across the river as a picture.

While gathering and recording their data, students begin to notice recurring patterns. Some students notice a pattern while working with the manipulatives and use their recorded data to check their idea of the pattern. In one class, for example, two girls quickly exclaimed to their teacher, "We found the pattern." Using the manipulatives, they showed their teacher and explained, "First you send two children over, next you bring one child back, then you send an adult over, and the other child brings the boat back. It doesn't matter how many adults you have, you just need two children. You can get any number of adults across even if it takes all day!" To explain how many total trips occurred for 8 adults, however, the two girls had to go through the process again, record all the trips, and then count the total number of trips. Other students look for the pattern in their recorded data (2 children across, 1 child back, 1 adult across, the second child back, 2 children across, 1 child back, and so on) and recognize that it would take four trips to get 1 adult across the river.

Pattern Recognition

Students try additional cases of the problem with different numbers of adults (step 2 in fig. 2). For some students, solving additional cases serves as a way to test and refine their understanding of the pattern (fig. 5), and for others, it provides additional opportunities to recognize a pattern (fig. 6).

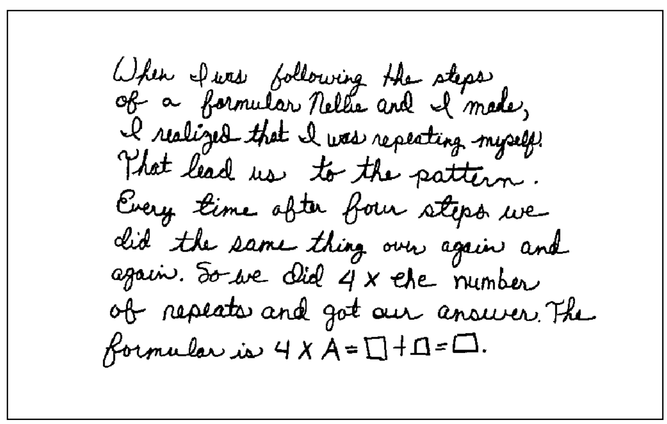
When I was following the steps of a formular Nellie and I made, I realized that I was repeating myself. That lead us to the pattern. Every time after four steps we did the same thing over again and again. So we did 4 x the number of repeats and got our answer. The formular is 4 x A = □ + □ = □.

Fig. 5. One student explains how she and her partner used the pattern to solve the problem and generate a formula.

In the last problem of this step, students are asked to explore how many trips it would take to get 100 adults across the river. At this point, some students do not use the manipulatives to get a solution; instead these students substitute 100 into their understanding of the pattern to calculate a solution (fig. 7). However, other students have been using the manipulatives to solve all the cases up to this point. One student's dilemma is captured in his comment, "On the paper it said what would be the answer if there were 100 grown-ups, and I thought we had to do the problem [for 100 grown-ups]. I hope we don't have to do that problem!" Asking students to solve the problem for a very large number of adults motivates them to look for a generalized method for solving the situation on the basis of patterns they saw in simpler cases rather than act out a very large case.

Students use multiple representations of the data in their search for a generalized statement of the pattern. If students are stuck, teachers may suggest that they represent the data in a table or a graph. Students have learned to read data from tables and graphs in introductory lessons in the unit; thus, the representations can help them find the patterns.

Generalizing

In the final step of the task (step 3 in fig. 2), students develop a generalized method for figuring out how many trips it would take to get everyone across the river, given any number of adults. Students articulate a generalized rule from their pattern in a way that they feel most comfortable—by using words, diagrams, their own invented symbols, or equations—and explain it by relating it to the original situation (figs. 8 and 9).

2 children go over
1 child rows back
1 adult goes over
1 child rows back

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2 Children row over
1 Child rows back
1 adult rows over
1 Child rows back

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2 children row over
Repeats Etc...

Fig. 6. Some students begin by writing things out but then recognize the pattern.

The most important aspect of this step is for students to describe how their generalization relates to the physical situation; why do you "multiply the number of adults by 4" and "add 1 for the last trip"? When students can answer these questions, they have gained an important understanding of how to use algebraic thinking to model a concrete situation.

From generalizing the pattern, students come to understand the power of algebraic thinking. As one student commented, "There was a pattern. Every four trips were the same. You can figure out how many trips it took by using a formula. To figure it out fast, you could use a formula." Another student wrote, "I think we try to find formulas so it will be easier to get the problems done. Formulas make problems easy to solve. It's very helpful." A third student wrote, "Sometimes when you think you found an equation, what you really found is the steps you took to get the answer in a shortened way." Seeing the power of algebraic thinking motivates students to engage in this kind of thinking when presented with similar situations.

Benefits of a Whole Unit Dedicated to this Approach

As in the "Crossing the River" example, the other lessons in the unit employ the same approach: students are given a contextualized problem, which they then solve through an investigative process. For example, in "Painted Rods" students are told of a company that produces painted rods of varying lengths by using a paint-stamping machine (see fig. 10). The paint stamp marks exactly one square of the rod each time. Students are asked, "How many paint stamps does it take to paint all the faces of different lengths of rods?"

All you have to do is multiply the number of adults by 4 and add 1 using the rule we talked about in school.
100 a 2C = 401
4×100 = 400+1 (the last child)
=401

Fig. 7. One student used a generalized understanding of the pattern to solve the problem for 100 adults.

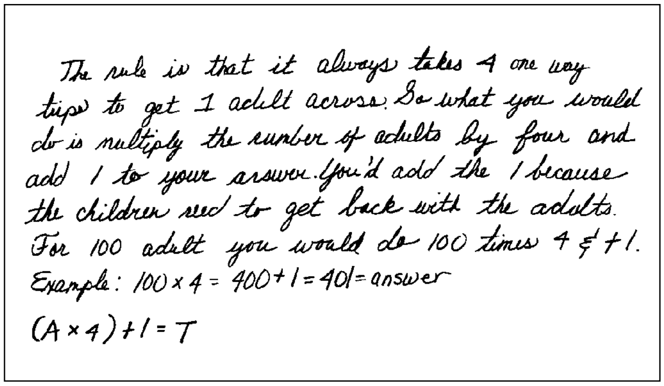
The rule is that it always takes 4 one way trips to get 1 adult across. So what you would do is multiply the number of adults by four and add 1 to your answer. You'd add the 1 because the children need to get back with the adults. For 100 adult you would do 100 times 4 & +1.
Example: 100 x 4 = 400 + 1 = 401 = answer
(A x 4) + 1 = T

Fig. 8. This student articulated the generalized rule in symbols and explained how it relates to the original situation.

Following the investigative process, students use manipulatives to construct models of the rods. A single cube serves as the paint stamp as students either physically emulate or visualize painting rods of lengths 1 to 10. As students solve the problems for these rods, they collect numerical data, organize them in a table, and use their table to search for a pattern in the numbers. Next, students find the number of stamps required to paint rods of length 12, 25, and 100. Solving such large cases as a rod of length 100 motivates students to find the general relationship between the length of a rod and the number of stamps required to paint it. Students then use their method of choice—words, diagrams, or their own invented symbols—to represent their generalized understanding of the pattern as a rule. Finally students explain their rule in terms of the original situation.

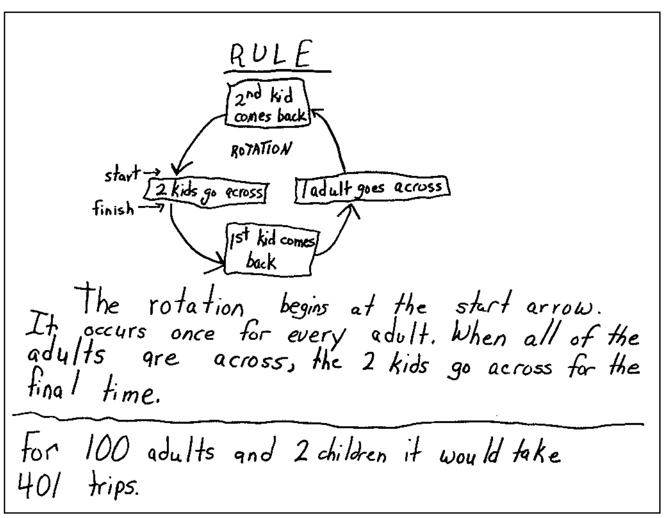

Fig. 9. Another student drew the generalized rule and explained how it relates to the original situation.

To extend their thinking, students consider what the input may have been for a given output. For example, in "Painted Rods" students are asked to use their understanding of the pattern to explore the question "If it takes 86 stamps to paint the rod, how long is the rod?" Another way that students' thinking is extended is by asking them to solve a similar, but more challenging, problem. After completing "Crossing the River," for example, students are asked to solve the same problem with a different number of children. "What happens to the pattern if we have a different number of children? (a) 8 adults and 3 children or (b) 2 adults and 5 children?" Similarly, after "Painted Rods," students are asked to solve the problem for rods of double width.

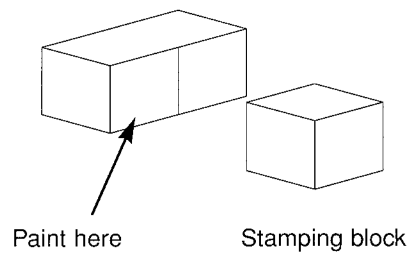

Fig. 10. Drawing of problem in "Painted Rods"

Together the lessons in the Patterns in Numbers and Shapes unit afford students multiple opportunities to learn the investigative process of pattern seeking, pattern recognition, and generalization. Students engage in the process in each lesson, and, consequently, over the course of the unit they internalize the approach to solving situations involving patterns. Students' understanding of and ability to use the process is evident in the strategies and skills for pattern seeking that they exhibit; the tools of tables, graphs, and verbal rules that they use for describing patterns; and the generalized rules they articulate. Thus, in the future when students are presented with a problem involving patterns, they have strategies for approaching the problem.

The unit also has a positive effect on students' perception of their ability to generalize a rule from a concrete situation, that is, to think algebraically. As students follow the process, they are able to find a pattern and express it as a generalized rule. One student wrote, "Finding formulas is exciting. It's like artifacts to an archaeologist." Finding a rule makes students feel successful, and, consequently, they begin to see themselves as being capable algebra students. Teachers who have taught the unit [in a field test] reported, "[Students] are confident.... They think they are good at it...." "They think they are smart. They know that their big brothers and sisters do algebra." Students' increased confidence in their ability to engage in algebraic thinking ultimately contributes to a positive attitude about algebra. Their positive attitude in conjunction with the use of an investigative process and their new understanding of the power of algebraic thinking give sixth-grade students a solid foundation on which to build a formal understanding of algebra throughout the middle grades.

Reference

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: The Council, 1989.