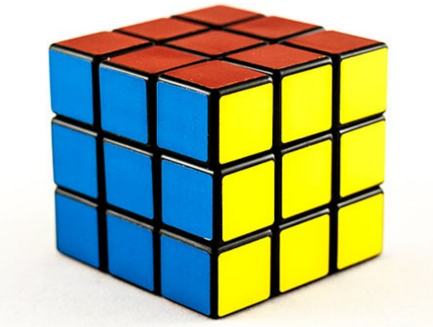
What's Mathematical About It?

In 1974, Erno Rubik designed a famous toy that has since captured the imagination of millions of people around the world. You've probably all seen this famous toy, the Rubik's Cube, at one point or another. Many of you might even have spent some frustrating hours trying to solve the cube puzzle – frantically rotating it's sides, trying to align it's colours. Some of you might even have emerged triumphant. For those of you who are not so familiar with the toy, below is an image of an untouched Rubik's Cube. After rotating it's faces for a few minutes, you find that the colours get completely jumbled, and there is no obvious way of returning to the original, pristine state. The challenge of the Rubik's Cube is to return it to it's original state.



An untouched Rubik's Cube

We often hear people talk of how "mathematical" the toy is. In fact, it's not uncommon to hear people describe toys, a piece of music, a painting or a design as mathematical – 'it's so mathematical', they say – in a tone that makes one want to replace the word mathematical with 'beautiful', or better still, 'beautiful and mysterious'. Have you ever wondered what they mean? What, really, is mathematical about the Rubik's Cube? What is it, that has prompted mathematicians to write research papers analysing the mathematical properties of this seemingly innocent toy? Surely not just the fact that it is a regular polyhedron! Erno Rubik himself was no mathematician. Born in the air-raid shelter of a Budapest hospital during World War II to a poet and an aircraft engineer, Rubik studied architecture and later taught interior design. He probably didn't anticipate the popularity of his invention. By 1982, 'Rubik's Cube' was a household term, and became part of the Oxford English Dictionary. More than 100 million cubes have been sold worldwide.[1]

At first glance, there is not much that seems mathematical about the Cube – there are no numbers involved, no equations to solve. But what if we asked questions like "How many possible configurations does the cube have?" or "What are the steps that would lead us back from this jumbled state to the original configuration?" We would have to turn to mathematics to answer these questions. Looking at the mathematics that describes the Rubik's Cube takes us to a fascinating field of study – one that deals with symmetry. In fact, most often, what is common to a "mathematical" piece of art or music or a toy like the Rubik's Cube, is symmetry. So before we see what symmetry has to do with mathematics, let us understand what we mean by it and how we 'measure' it.

We describe something as symmetrical when it looks the same from more than one point of view. For example, human beings are said to (well, almost) have bilateral symmetry since our left side is similar to our right side, and we (more or less) look the same in the mirror as we do face-to-face. A ceiling fan on the other hand, has 3-fold rotational symmetry – If you rotate it a third of a circle, it appears the same as it did before it was rotated.

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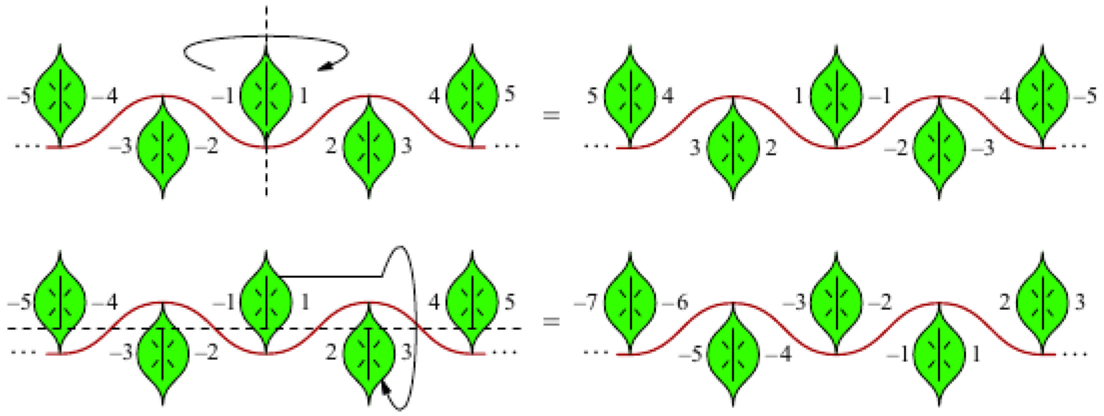
1 From Joyner, D. (2002). *Adventures in group theory: Rubik's Cube, Merlin's machine, and other mathematical toys*. Baltimore: Johns Hopkins University Press.

Take a look at this wall drawing as well. Gliding and rotating it or flipping it horizontally doesn't change how it appears – it seems unchanged at each position. Such patterns that repeat in one dimension are called *frieze* patterns.



Horizontal flip

Glide and rotate

So how do we measure the symmetry of these objects and images? Since we're more concerned with shape and structure than things like colour, in stead of speaking about 'looking the same' we think of those situations in which the object or image would *fit into the same space*. Measuring symmetry involves making a list of all the ways in which you can re-arrange an object so that it occupies the same space as it did originally.

When we talk of the Rubik's cube as well – we're looking at rearrangements of the cubes parts. The cube itself is so designed that any move we make retains the original shape of the cube while rearranging the parts. So studying the cube and its configurations would mean studying its symmetries.

You're probably still wondering where the mathematics is in all of this. It turns out that the collection of these rearrangements we're speaking about, form what mathematicians call a *group*. All these collections follow a fixed set of rules and behave in similar ways. Studying the underlying structure of these groups and classifying them in meaningful ways enables us to predict the ways in which they behave.

Let's make some observations about the Rubik's Cube and see if we can extract at least an informal notion of a group.

The moves allowed in the Cube are restricted by its design – in fact, we can say that it has only 6 six moves – rotating each of its 6 faces 90 degrees clockwise. Of course, one can rotate a face 180 degrees, or even 90 degrees counter-clockwise, but this is the same as rotating a face 90 degrees clockwise twice and thrice respectively. And rotating it 360 degrees is the same as a 'do nothing' move. So, from this we make our first important observation -

- ***There is a predefined list of actions (moves) that never changes***

Now, whatever move you make, you can always undo it by rotating it the other way. So,

- ***Every action is reversible***

The Rubik's Cube is not a game of chance unlike several card games which depend on the hand you've been dealt, or games with dice where you're unaware what number will be thrown up. Here, rotating a face has a predictable outcome – not dependent on skill or luck in any way. Thus

- ***The outcome of every action can be determined in advance***

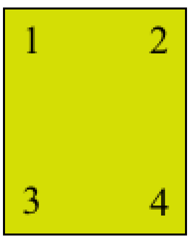
And finally, any sequence of moves is like an action in its own right, formed by a combination of the basic moves we listed in our predefined list of actions. There is no restriction on the order in which moves can be combined. Any sequence of moves puts the cube back into a position in which any other sequence of moves can be applied. So we say that

- ***Any sequence of consecutive actions is also an action***

We can make several more observations about the cube, but it is the mathematical consequences of these four observations that we are interested in – they suffice to describe the symmetry of the Rubik's Cube. If we reformulated these observations as rules, then we can say that in an informal way, *a **group** is a collection of actions that follows these four rules.*[2]

How do these rules help us understand something about the cube? How would they help us solve it? Given a jumbled cube, it is almost always near impossible to see a sequence of actions that will help you get to the solved cube. What would be extremely helpful to someone lost in the confounding maze of the cube's configurations, would be a map that would help direct you from where you are to where you want to go (a path to get to a solved cube). Making such a map would mean making a map of every possible configuration and how each of these configurations relate to one another. In other words, we will have to look at all possible sequences of actions (all actions generated by a combination of the 6 listed actions), how they combine and how they relate to one another. We have already seen that the moves of the Rubik's Cube form a group. The exhaustive list of these moves generated by the 6 basic moves would be the *elements* of the group. And the map we make would be a map of the group formed by *all possible combinations of moves of the cube.* There are several books and papers on the Rubik's Cube that provide us with this information. But as you can well imagine, such a map would be enormous. As it turns out, the size of the group is 43252003274489856000 – a number that is difficult even to read! So making a map of the Rubik's Cube group is a difficult task to say the least, and for this reason, beyond the scope of this article. But that needn't stop us from understanding the ideas involved – we can instead look at a simpler example of a group and get a taste of what a map of it would look like.

Consider a rectangle as in the picture below. Looking at its group of symmetries would mean we have to first look at all possible moves we can make so that the rectangle fits into the same space as it does in the figure. You can think of it as a mattress in a bed – and ask – "What are the ways in which one can re-arrange the mattress so that it still fits in the bed frame?"

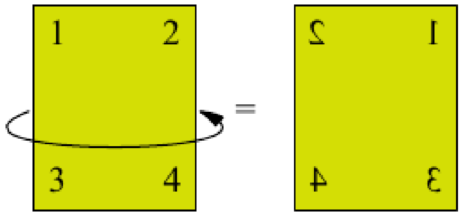


A rectangle with its corners numbered

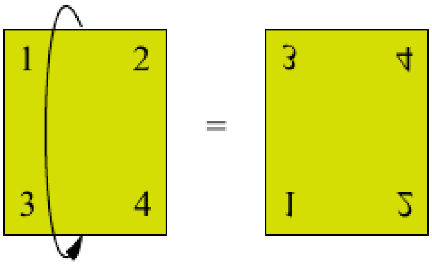
A rectangular mattress with its corners numbered

Two possible moves that strike us immediately are to flip the rectangle horizontally and to flip it vertically –

2 These rules and definitions have been adapted directly from those used by Nathan Carter in his book *Visual Group Theory*

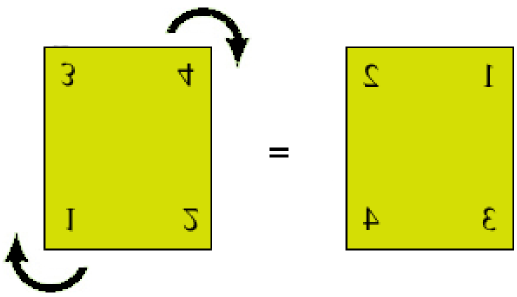


horizontal flip (H)



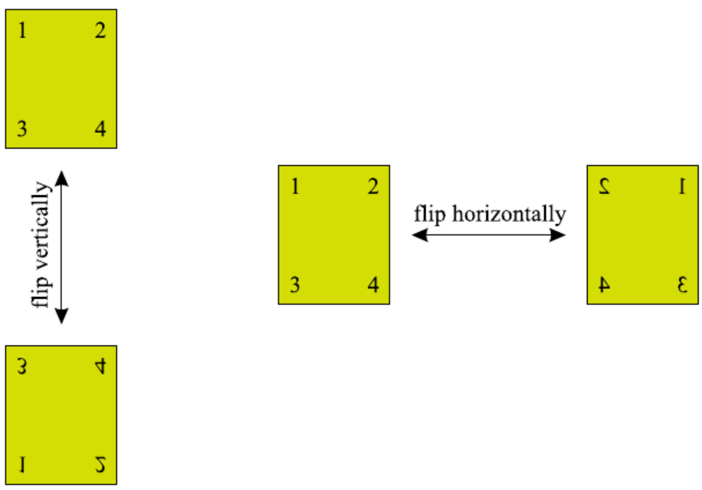
vertical flip (V)

One could also rotate the mattress 180 degrees clockwise. (Rotating it 180 degrees counter-clockwise is equally valid, but it amounts to the same as rotating it 180 degrees clock-wise)

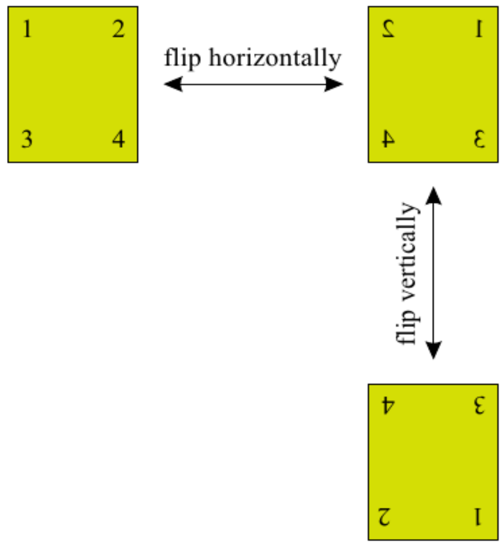


rotation by 180 (R)

Now we must look at combinations of these moves. Notice that by performing 2 horizontal flips, we return to the original position. Similarly, combining 2 vertical flips also gets us back to the original position. So HH = VV = N (where N is the 'do nothing' move). We can show this using arrows as follows

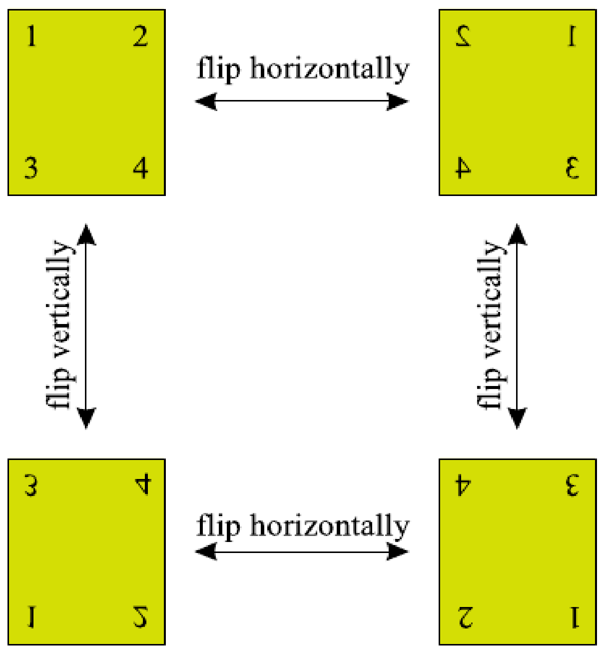


Combining a horizontal flip followed by a vertical flip results in this -



Performing a vertical flip followed by a horizontal flip would result in the same. What's more, this is the same as performing a rotation of 180 degrees. (Look at the figure on the previous page) So HV = VH = R. Since R is nothing but a combination of H and V, we needn't worry about it. This means that there are only two moves that we are concerned with – the horizontal flip (H) and the vertical flip (V).

First lets take a moment to verify that these moves follow the four rules of a group we defined earlier. Once you have done this, then in order to map the group of symmetries, we have to look at all possible combinations. By doing this, we are listing all the elements of the group, and also looking at how they relate to one another. This can be nicely represented by the following diagram.



Full map of configurations of the rectangle

Of course, just as in the case of the Rubik's Cube, one can combine moves repeatedly in any order. And you're probably wondering why these endless combinations wouldn't form elements of the group of symmetries as well. Try out a few combinations for yourself and you will find that any combination will result in one of the four positions in the diagram above that are equivalent to performing any one of the the four moves which are – *do nothing (N), flip vertically (V), flip horizontally (H), flip horizontally and flip vertically (HV).* For example, imagine the move *horizontal flip – vertical flip – vertical flip – horizontal flip.* From the map above, it is clear that this would amount to doing nothing. In symbols this means HVVH = N (see the picture given above).

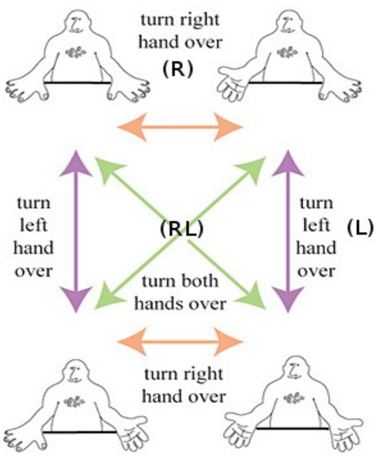
Try and make an informal argument for why this is true.

So the diagram above is actually a complete map of the *group of symmetries of the rectangle*! Thinking of it as a puzzle like the Rubik's Cube – with the challenge being to return the rectangle to it's original configuration from the situation you are in, the diagram serves as a map that can guide you to solving the puzzle.

We can make a table for such a map using symbols to show how the elements of the group combine – just like we make multiplication tables in primary school -

	N	**H**	**V**	**HV**
N	N	H	V	HV
H	H	N	HV	V
V	V	HV	N	H
HV	HV	V	H	N

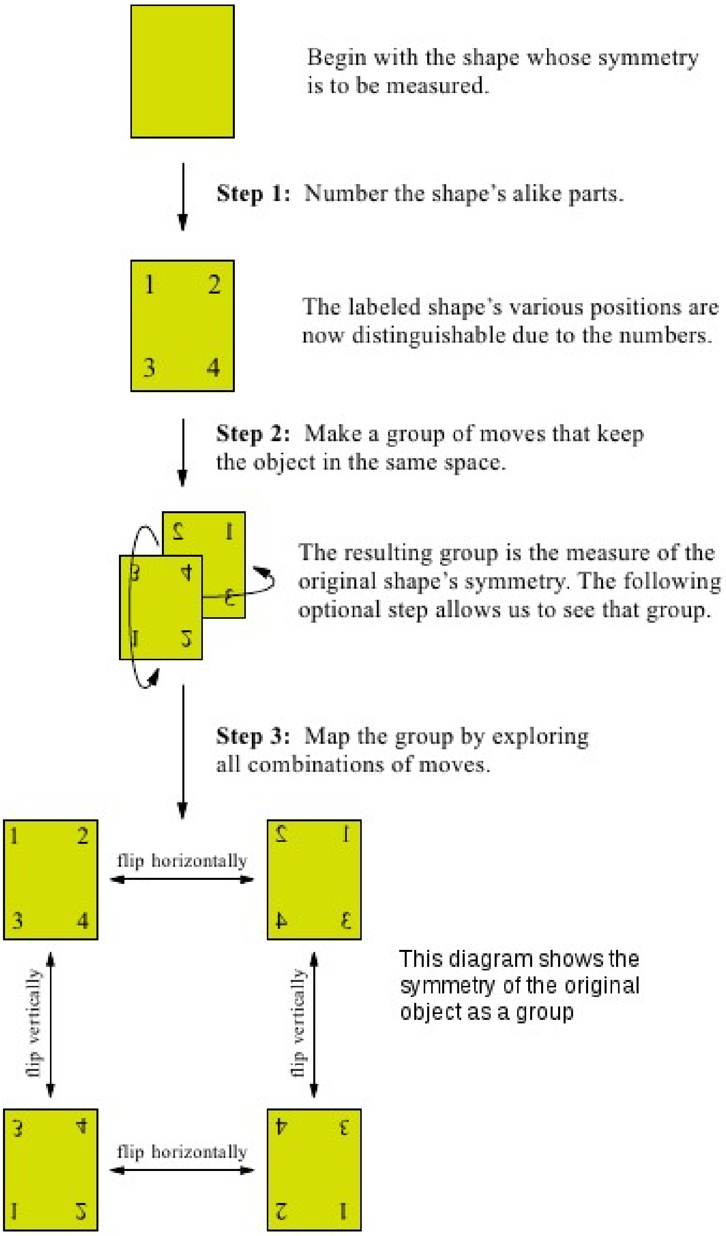
This group – the group of symmetries of a rectangle – shows up in some unexpected places, from the symmetry of water molecules to the logic of a pair of electrical switches. As Steven Strogatz writes "That's one of the charms of group theory. It exposes the hidden unity of things that would otherwise seem unrelated … like this anecdote about how the physicist Richard Feynman got a draft deferment. The army psychiatrist questioning him asked Feynman to put out his hands so he could examine them. Feynman stuck them out, one palm up, the other down. "No, the other way," said the psychiatrist. So Feynman reversed *both* hands, leaving one palm down and the other up. Feynman wasn't merely playing mind games; he was indulging in a little group-theoretic humor." Consider all the possible ways he could have held out his hands – it can be shown in a diagram like this:



Does this seem familiar? It follows the same pattern as that of the group of symmetries of a rectangle! Make a multiplication table (using the symbols R for turn right hand over, L for turn left hand over, RL for turn both (right and left) hands over and N for 'do nothing') and compare it to the table we made for the rectangle. Interesting isn't it, how we find a similar pattern in two such vastly different and seemingly unrelated situations?

Group theory allows us to see these connections in a surprising variety of contexts. From the structure of number domains and the shape of molecules and crystals to frieze patterns found on the walls of monuments, groups are used to describe symmetry in all sorts of places. The beautiful thing about group theory is that it gives us a way of recognizing similar patterns in unexpected places – by extracting the structure of a group that describes a particular object or pattern, we are confronted by the hidden unity that Strogatz spoke of, leaving us with a sense of wonder at how mathematics explains phenomena around us in such unreasonably effective ways. For example, despite the infinite possibilities we seem to have to create frieze patterns, group theory tells us that any frieze pattern can be described by one of just 7 different groups! Similarly all wallpaper patterns – which are patterns that extend in two dimensions in stead of one – belong to one of just 17 possible groups! The grand palace Alhambra in Spain is supposed to have examples of patterns belonging to each of these 17 wallpaper groups.

Symmetry has recently found a place in the primary school curriculum. In a vague sense, I suppose we all knew that it had *something* to do with mathematics, but it might not have been clear what this was exactly. Looking at the group of symmetries of a rectangle has given you a taste of what symmetry has to do with mathematics. It has also acquainted you with the technique of finding a group that describes an objects symmetry. As a quick recap, the diagram below defines the steps in which this can be done, using the rectangle as an example.



Group theory is not introduced until the second or third year of graduation. So whether the inclusion of symmetry in the primary school curriculum is justified, I do not know. But at least the next time you hear someone look at something apparently symmetrical and say 'this is very mathematical' you can smile to yourself and say, 'I think I know what you mean.'

This piece was inspired by an article called Group Think *by Steven Strogatz that appeared in The New York Times some years ago. It draws heavily on, and has adapted excerpts of , the book* Visual Group Theory *by Nathan Carter. Images are credited to both references. For a formal study of groups, here are some standard references - Artin, M. (2011). Algebra. 2nd ed. Boston: Prentice Hall., Dummit, D. Steven, & Foote, R. M. (2004). Abstract algebra. 3rd ed. Hoboken, NJ: Wiley., Herstein, I. N. (1996). Abstract algebra. 3rd ed. Upper Saddle River, N.J.: Prentice-Hall.*

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Carter, N. C. (2009). *Visual group theory*. Washington, D.C.: Mathematical Association of America.

Strogatz, S. (2010, May 2). *Group Think*. Retrieved from http://opinionator.blogs.nytimes.com/2010/05/02/group-think/

Joyner, D. (2002). *Adventures in group theory: Rubik's Cube, Merlin's machine, and other mathematical toys*. Baltimore: Johns Hopkins University Press.